MA 681, Spring 2011

Assignment 1.

Basic techniques. Möbius transformations. Differentiability

I prefer that you submit this assignment by Wednesday, February 9th. However, if you are somehow delayed, take your time (just don't get overwhelmed by homeworks piling up.)

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

(1) Represent the following complex numbers in trigonometric form: (a) $1+i$, (b) $-1+i$, (c) $-1-i$, (d) $1+i\sqrt{3}$, (e) $-1+i\sqrt{3}$, (f) $\sqrt{3}-i$.

(2) Calculate
  (a) $\frac{1+i\tan\alpha}{1-i\tan\alpha}$ (where $\alpha \in \mathbb{R}$),
  (b) $\frac{(1+i)^{2011}}{(1-i)^{2009}}$.

(3) Calculate
  (a) $(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$,
  (b) $(a+b)(a+b\omega)(a+b\omega^2)$,
  where $\omega = -\frac{1}{2}+\frac{1}{2}\sqrt{3}\cdot i$.

(4) Prove the identity $|z_1+z_2|^2+|z_1-z_2|^2 = 2(|z_1|^2+|z_2|^2)$. (By the way, what is the geometric interpretation of this identity?)

(5) Prove that any complex number of absolute value 1 (except for $z=-1$) can be represented as

$$z = \frac{1+it}{1-it},$$

where t is a real number (Hint: compare to 2a. Alternative hint: find image of the real line under Möbius transformation $w = \frac{1+iz}{1-iz}$).

(6) Use the fact that $1+\cos\alpha+\cos 2\alpha+\cdots+\cos n\alpha = \operatorname{Re}(1+z+z^2+\cdots+z^n)$, where $z=\cos\alpha+i\sin\alpha$, to find a trigonometric expression for $1+\cos\alpha+\cos 2\alpha+\cdots+\cos n\alpha$.

(7) $z\overline{z}-E\overline{z}-\overline{E}z+D=0$ is an equation of a circle ($E\in\mathbb{C}, D\in\mathbb{R}$). Find its center and radius.

(8) Find the images of the following curves under transformation $w=1/z$:
  (a) The family of circles $x^2+y^2=ax$ ($a\in\mathbb{R}$).
  (b) The family of parallel lines $y=x+b$ ($b\in\mathbb{R}$).
  (c) The family of lines $y=kx$ passing through the origin ($k\in\mathbb{R}$).

(9) Find M obius transformation that carries points $-1, i, 1+i$ into
  (a) $0, 2i, 1-i$,
  (b) $i, \infty, 1$.

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(10) Find the images of the following domains under the indicated Möbius transformations:
   (a) The quadrant $x > 0,\ y > 0$ if $w = \frac{z-i}{z+i}$.
   (b) The half-disc $|z| < 1$, $\operatorname{Im} z > 0$ if $w = \frac{2z-i}{2+iz}$.
   (c) The strip $0 < x < 1$ if $w = \frac{z}{z-1}$.
   (d) The strip $0 < x < 1$ if $w = \frac{z-1}{z-2}$.

(11) Show that the function $f(z) = z \operatorname{Re} z$ is differentiable only at the point $z = 0$, and find $f'(0)$.

(12) Let $z_0 \neq 0$ and let $f(z) = \ln r + i\Phi$, where $r = |z|$, $\Phi \in \operatorname{Arg} z$, and Φ is chosen so that f is continuous in a neighborhood of z_0. Prove that f is differentiable in a neighborhood of z_0.